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(Student Paper) Efficient Uncertainty Quantification in Computational Fluid-Structure Interactions

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In this paper a Two Step approach for efficient uncertainty quantification in computational fluid-structure interactions is followed. In Step I, a Sensitivity Analysis is used to efficiently narrow the problem down from multiple uncertain parameters to one parameter which has the largest influence on the solution. In Step II, for this most important parameter a more advanced uncertainty quantification method is employed to obtain the stochastic response of the solution. In order to find an efficient method for this, Monte Carlo simulation, the Askey Polynomial Chaos method, the Stochastic Collocation method, and the Piecewise Interpolated Sampling method are compared for the linear piston problem. In addition, the Piecewise Interpolated Sampling method, which uses piecewise cubic polynomials to interpolate between samples directly taken from the parameter space, is presented. This leads to better coverage of the parameter domain, compared to the Stochastic Collocation method, when the distribution has tails like the exponential and normal distribution. Finally, the efficiency of the Two Step approach is demonstrated for the linear piston problem with an unsteady boundary condition. A speed-up factor of $\mathcal{O}(10^2)$ is obtained compared to a full uncertainty analysis for all parameters.

I. Introduction

There is an increasing interest in uncertainty analysis applied to computational fluid-structure interactions, since the influence of inherent physical uncertainties can no longer be neglected. In the past decades numerical errors have been decreased due to more accurate algorithms and more powerful computing facilities to such an extend, that uncertainties are now the limiting factor in obtaining reliable results. When fluid-structure interaction is concerned, small uncertainties in the input parameters can lead to unreliable solutions after long time integration. Therefore, to obtain more reliable results, uncertainty quantification is necessary.

Deterministic computations for fluid-structure interaction problems are already computationally intensive due to the dynamic coupling between the fluid and structure. Uncertainty quantification requires an additional method to obtain the stochastic properties, this leads to an increase in the computational work compared to the deterministic case. Especially for multiple uncertain parameters the amount of work increases rapidly. Efficiency of the uncertainty quantification is, therefore, of great importance.

In this paper the efficiency of uncertainty quantification for multiple uncertain parameters is increased by following a Two Step approach. In Step I, a Sensitivity Analysis is employed to identify the most important parameter[1] of the problem. In Step II, the stochastic response of the solution is obtained for the most important parameter using an more advanced method. In the second step only methods are considered which result in the probability distribution of the solution based on the input distribution of the uncertain

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parameter. Since computational fluid-structure interaction problems are computationally intensive, the second step has to be performed as efficient as possible.

Sensitivity Analysis is an efficient way to identify the most important uncertain parameter of the problem. Sensitivity Analysis compares the scaled sensitivity derivatives only. No stochastic properties of the input parameters are taken into account. Sensitivity derivatives give the sensitivity of the solution with respect to a parameter. They are computed using the continuous sensitivity equations.[20] The parameter with the largest scaled sensitivity derivatives has the largest influence on the solution and is identified to be the most important parameter of the problem.

In Step II, the stochastic response of the solution is computed using a more sophisticated uncertainty quantification method. In literature, among others, the following uncertainty quantification methods can be found: Monte Carlo simulation, the Askey Polynomial Chaos, and the Stochastic Collocation method.

Monte Carlo simulation is the most natural way to obtain the stochastic response. It was already applied in 1901 by Kelvin.[10] Typically in the order of thousands of samples are required to obtain reasonable accuracy. Techniques exist to minimize the number of samples, like stratified sampling (Latin Hypercube) or variance reduction. An advantage of Monte Carlo simulation is that the method is non-intrusive, a disadvantage is the amount of computational work.

The original Polynomial Chaos method was developed by Ghanem and Spanos[6] based on the Homogeneous Chaos theory of Wiener.[23] They used Hermite polynomials to obtain exponential convergence for Gaussian random variables and show the successful application to computational mechanics problems.[4,5] An important extension of this framework is the Askey Polynomial Chaos by Xiu and Karniadakis.[25] The Askey scheme provides orthogonal polynomials to obtain exponential convergence for certain standard probability distributions. The Askey Polynomial Chaos is successfully applied to several problems including fluid-structure interactions.[26] Recently, the Polynomial Chaos framework has been extended to obtain exponential convergence for arbitrary distributions[22,24] using numerical techniques to construct an optimal set of orthogonal polynomials. An advantage of the Polynomial Chaos method is the exponential convergence with respect to the polynomial order, a disadvantage is the intrusiveness due to the coupled system of equations that has to be solved. Recently, a non-intrusive polynomial chaos[9] method is developed to simplify the implementation using existing solvers.

The Stochastic Collocation method was developed by Mathelin and Hussaini[18,19] to reduce the computational work of the Askey polynomial chaos method when non-linearities are involved. The Stochastic Collocation method is based upon a transformation between the stochastic space and an artificial space. The probability distribution of the uncertain parameter serves as the basis of the transformation. Collocation points are chosen from this artificial space, after which Lagrange interpolation is used to approximate the response function. The method is successfully applied to a quasi-1D nozzle with uncertain inlet conditions and nozzle shape. Mathelin and Hussaini showed a substantial decrease of computing time using the Stochastic Collocation method compared to the Polynomial Chaos method.

In addition, the Piecewise Interpolated Sampling method is introduced in this paper for application in the second step. This method uses a piecewise cubic interpolation as opposed to the Stochastic Collocation method, which use global polynomials to approximate the stochastic response of the solution. A disadvantage of global polynomials is that a small fluctuation over a small portion of the interval can induce large fluctuations over the entire domain.[2] When long time integration is considered for input distributions with tails like the exponential and normal distribution the stochastic response can show large fluctuations in the tail areas. These fluctuations lead to bad approximations in the smooth parts as well. Therefore, a piecewise cubic interpolation is employed to approximate the response. The piecewise interpolation is performed between samples, which are taken uniformly distributed directly from the parameter domain to make sure that the complete parameter domain is covered evenly. Areas that deform heavily (e.g. tails of the distribution) do not lead to large oscillations in the entire domain.

The efficiency of the uncertainty quantification methods used for Step II is compared by looking at the error convergence with respect to the computational work. The computational work is expressed in the number of times a deterministic system is solved. The linear piston problem[16] is employed as test problem, since it is a comprehendable problem which possesses all aspects of fluid-structure interaction. Lin et al.[13] show that the length of the time integration is of critical influence on the performance of the Askey Polynomial Chaos method with respect to Monte Carlo simulation. Therefore, the efficiency is compared for both short and long time integration. Three distribution types are investigated, based on the parameter domain. They are the uniform distribution on $[a, b]$, the exponential distribution on $[a, \infty)$, and the normal distribution on

$(-\infty, \infty)$. These distributions are chosen from the Askey scheme to obtain an optimal Polynomial Chaos respresentation.

Finally, the Two Step approach is demonstrated for the linear piston with an unsteady boundary condition. First a Sensitivity Analysis is performed for the four uncertain parameters of the test problem. For the most important parameter the most efficient method is used to find to stochastic piston position. A speed-up factor of $\mathcal{O}(10^2)$ is found compared to a full uncertainty analysis for all four parameters.

This paper starts with a brief review of the uncertainty quantification methods that are applied, i.e. for Step I: the Sensitivity Analysis and for Step II: Monte Carlo simulation using stratified sampling, Askey Polynomial Chaos, Stochastic Collocation. Section III introduces the Piecewise Interpolated Sampling method which can be used in Step II. The efficiency of the methods of Step II is compared for a computational fluid-structure interaction problem for short and long time integration in section IV. A demonstration of efficient uncertainty quantification using the Two Step approach is shown in section V for a linear piston with an unsteady boundary condition. Section VI summarizes the results and conclusions are drawn.

II. Two step approach for efficient uncertainty quantification

Since the propagation of probability distributions for multiple uncertain parameters is computationally intensive a two step approach is followed. The first step consists of a Sensitivity Analysis, which is performed to identify the most important parameter of the problem. After that in the second step the uncertainty of the identified parameter is propagated using an efficient more advanced method which results in the stochastic response of the solution based on the input distribution of the uncertain parameter. For Step II one can use either Monte Carlo simulation, the Polynomial Chaos, Stochastic Collocation, or the Piecewise Interpolated Sampling method. In section IV the efficiency of these methods is compared.

In this section the methods are explained using the following general differential equation

$$\mathcal{L}u(\mathbf{x}, t, \theta) = S(\mathbf{x}, t, \theta), \tag{1}$$

where $u(\mathbf{x}, t, \theta)$ is the solution and $\mathcal{L}$ is a differential operator which contains space and time differentiation and can be nonlinear. For example $\mathcal{L} = \partial/\partial t + u\partial/\partial x$ results in $\mathcal{L}u = u_t + uu_x$. The solution $u(\mathbf{x}, t, \theta)$ is a function of space $\mathbf{x}$, time t and the random event $\theta \in [0, 1]$. $S(\mathbf{x}, t, \theta)$ is a space and time dependent source term which can also depend on the random event θ. The random event θ is introduced by an uncertain parameter $p(\theta)$.

A. Step I: Sensitivity Analysis to obtain the most important parameter

Sensitivity Analysis is based on the sensitivity derivative. The sensitivity derivative is a measure for the sensitivity of the solution with respect to a parameter. A large derivative means a large sensitivity. The sensitivity derivative can be used for calculating an uncertainty interval. In literature this is done for different flow problems: laminar flow,[14] turbulent flow with and without heat transfer.[21,3] Another possibility is to use the sensitivity derivatives in perturbation methods to obtain low order estimates of the mean and variance.[12] In this paper sensitivity derivatives[20] are used to identify the most important uncertain parameter in a particular physical system. Sensitivity is used here as an efficient way of reducing the amount of uncertain parameters.

The sensitivity derivative of the solution $u(\mathbf{x}, t, \theta)$ with respect to the uncertain parameter $p(\theta)$ is defined as

$$u_p = \frac{\partial u(\mathbf{x}, t, \theta)}{\partial p(\theta)}. \tag{2}$$

The sensitivity derivative can be computed in several ways. Common methods are, among others, finite differencing, the complex step method,[15] automatic differentiation,[17] and the continuous sensitivity equation approach.[21] Here the continuous sensitivity equation approach is used since the implementation is straightforward and the solution is exact. The continuous sensitivity equation solves for the sensitivity derivatives directly, it is the derivative of the original equation with respect to the uncertain parameter. Two equations have to be solved, namely

$$\mathcal{L}u(\mathbf{x},t,\theta;\mu_p) = S(\mathbf{x},t,\theta;\mu_p) \tag{3}$$

$$\frac{\partial}{\partial p}\Big\{\mathcal{L}u(\mathbf{x},t,\theta;p(\theta)) = S(\mathbf{x},t,\theta;p(\theta))\Big\}\Big|_{p(\theta)=\mu_p}, \tag{4}$$

where μ_p is the mean of the parameter $p(\theta)$. The first is the original differential equation of the problem (1) using μ_p for $p(\theta)$. The second equation (4) is called the continuous sensitivity equation.

When more parameters are involved in the problem the sensitivity derivative with respect to each parameter is calculated. By scaling the sensitivity derivatives with the nominal values of the parameters all sensitivity derivatives have the dimension of $u(\mathbf{x},t,\theta)$ and can be compared. The nominal value is the value of the parameter which would be used for deterministic computations, in this case the mean value μ_p. If the solution depends on N parameters, the most important parameter is the maximum of

$$u_{p_1}\mu_{p_1}\ ,\ u_{p_2}\mu_{p_2}\ ,\ldots,\ u_{p_N}\mu_{p_N}. \tag{5}$$

Once the most important parameter is obtained by Eq. (5) the stochastic response of the solution can be computed based on the input distribution of the uncertain parameter.

B. Step II: Uncertainty quantification methods to obtain the stochastic response of the solution

In our search for the most efficient method for Step II, only methods are used which result in the complete probability distribution function of the solution. The methods for Step II increase the amount of work with respect to deterministic computations. Therefore, in case of the computationally intensive fluid-structure interactions, the method which is used to obtain the stochastic response also has to be efficient. Basically, the methods can be divided in three categories. The first category are the sampling methods which result in the exact stochastic response, Monte Carlo simulation is such a method. The second category uses global polynomials to approximate the stochastic response, examples are the Polynomial Chaos and Stochastic Collocation method. The third category uses local polynomial approximations of low order to approximate the stochastic response, for example the Piecewise Interpolated Sampling method.

1. Monte Carlo simulation

Monte Carlo simulation is a natural way of quantifying uncertainties. For a number of values (samples) of the uncertain parameter the problem is solved deterministically. From the resulting set of solutions the stochastic properties are derived. For reasonable accuracy typically a high number of samples is required. Monte Carlo simulation was first applied by Kelvin[10] in 1901. Monte Carlo simulation works as follows:

1. Take a value from the domain [0,1], this is called sampling;
2. Calculate the corresponding value for the random variable using its distribution function;
3. Solve the problem like a deterministic problem;
4. Repeat this many (M) times and analyze the statistical properties of the set of solutions.

The sampling can be done in different ways. The original Monte Carlo simulation uses random sampling, with a convergence rate of $\mathcal{O}(M^{-1/2})$. Here stratified or Latin Hypercube sampling is used, the convergence is improved, but to $\mathcal{O}(M^{-1})$ only. That is why in practice thousands of samples are used, resulting in computational costs thousands times higher than the deterministic computation.

2. Polynomial Chaos

The Polynomial Chaos method results in a spectral representation of the stochastic response of the solution and high order approximations of the mean and variance. Based on the Homogeneous Chaos theory of Wiener[23] the original Polynomial Chaos method was developed by Ghanem and Spanos.[6] They used Hermite polynomials for Gaussian random variables and show the successful application to computational mechanics problems.[4,5] This framework was extended by Xiu and Karniadakis[25] to the Askey Polynomial Chaos. The Askey scheme provides orthogonal polynomials to obtain exponential convergence of the solution with respect to the polynomial order of the approximation for certain distribution. For example, the Askey scheme prescribes for uniformly, exponentially and normally distributed parameters Legendre, Laguerre and Hermite polynomials, respectively. Due to the chaos representation the input uncertainties can be expressed in two terms only. The choice of the optimal set of orthogonal polynomials leads to a weakly coupled system of governing equations and exponential convergence with respect to the polynomial order. The Askey Polynomial Chaos has been successfully applied to several problems, among others fluid-structure interactions.[26]

The Polynomial Chaos method starts with the polynomial chaos expansion of the solution $u(\mathbf{x}, t, \theta)$, with the infinite sum truncated at a finite number of $P + 1$ terms to find the following P^{th} order approximation

$$u(\mathbf{x}, t, \theta) \approx \sum_{i=0}^{P} u_i(\mathbf{x}, t)\boldsymbol{\Psi}_i(\boldsymbol{\zeta}(\theta)). \tag{6}$$

This expansion is a spectral expansion in the vector of random variables $\boldsymbol{\zeta}(\theta)$ with the random trial basis $\{\boldsymbol{\Psi}_i\}$. Equation (6) divides the random variable $u(\mathbf{x}, t, \theta)$ into a deterministic part, the coefficients $u_i(\mathbf{x}, t)$ and a stochastic part, the polynomials $\boldsymbol{\Psi}_i(\boldsymbol{\zeta}(\theta))$. The total number of expansion terms is $P + 1$, which is determined by the dimension n of the vector of random variables $\boldsymbol{\zeta}(\theta)$ (i.e. the number of uncertain parameters) and the highest order p of the polynomials $\{\boldsymbol{\Psi}_i\}$

$$P + 1 = \frac{(n + p)!}{n!p!}. \tag{7}$$

Substituting the polynomial chaos expansion (6) into the differential equation (1) results in

$$\mathcal{L} \sum_{i=0}^{P} u_i(\mathbf{x}, t)\boldsymbol{\Psi}_i(\boldsymbol{\zeta}(\theta)) \approx S(\mathbf{x}, t, \theta). \tag{8}$$

Here, the source term S has to be expanded as well, however for convenience in this notation it is omitted. To ensure that the truncation error is orthogonal to the functional space spanned by $\{\boldsymbol{\Psi}_i\}$ a Galerkin projection on each basis $\{\boldsymbol{\Psi}_k\}$ is applied

$$\left\langle \mathcal{L} \sum_{i=0}^{P} u_i(\mathbf{x}, t)\boldsymbol{\Psi}_i, \boldsymbol{\Psi}_k \right\rangle = \langle S, \boldsymbol{\Psi}_k \rangle, \qquad k = 0, 1, \ldots, P. \tag{9}$$

This deterministic set of $P+1$ coupled equations can be solved using standard numerical techniques.[7,8] Here a Block-Gauss-Seidel iterative method is employed to obtain the coefficients of the expansion. Since some iterations are required, the amount of work increases to several times $P + 1$. From Eq. (9) the coefficients $u_i(\mathbf{x}, t)$ are calculated and the stochastic solution $u(\mathbf{x}, t, \theta)$ is reconstructed using Eq. (6). The mean μ_u and the variance σ_u^2 of the stochastic solution can be determined using:

$$\mu_u = u_0, \tag{10}$$

$$\sigma_u^2 = \sum_{i=1}^{P} u_i(\mathbf{x}, t)^2 \left\langle \boldsymbol{\Psi}_i^2 \right\rangle. \tag{11}$$

These expressions follow from the definition of the mean and variance. The output of the Polynomial Chaos method is a spectral approximation of order P of the stochastic response and a high order approximation of the mean an variance.

3. *Stochastic Collocation*

A different spectral approach is the Stochastic Collocation method. It is derived from the Collocation or Pseudo-Spectral method. It was developed by Mathelin and Hussaini[18,19] to reduce the costs of the Askey Polynomial Chaos method in case of nonlinear equations. The Stochastic Collocation method utilizes collocation points for which corresponding values of the uncertain parameter are computed using the parameters distribution function. For each collocation point the problem is solved deterministically. The distribution function of the solution is reconstructed using global polynomial interpolation. The mean and variance are obtained using Gaussian quadrature. In case of the Stochastic Collocation method the distribution function of the random variable is projected from $[0,1]$ on the domain $[-1,1]$, which is called the α domain, by the linear transformation

$$\mathcal{F}_{\tilde{u}}\left(\tilde{u}\right) = 2F_u\left(u\right) - 1 \qquad \text{or} \qquad \alpha = 2\theta - 1, \tag{12}$$

where $\mathcal{F}_{\tilde{u}}\left(\tilde{u}\right)$ is the projected distribution function on the α domain $[-1,1]$ and $F_u\left(u\right)$ the distribution function on $\theta \in [0,1]$. The projected solution is $\tilde{u}(\mathbf{x},t,\alpha)$. The α domain is a stochastic space which is defined according to a standard domain of orthogonal polynomials $[-1,1]$. From the α domain N_p collocation points α_i are taken. The method proposed by Mathelin and Hussaini[18] uses N_p Gauss-Legendre quadrature points and Lagrange interpolating polynomials of order $N_p - 1$ for the function approximation. The projected stochastic solution $\tilde{u}(\mathbf{x},t,\alpha)$ is approximated by the following expansion

$$\tilde{u}(\mathbf{x},t,\alpha) \approx \sum_{i=1}^{N_p} \tilde{u}_i(\mathbf{x},t) h_i\left(\alpha\right), \tag{13}$$

with $\tilde{u}_i(\mathbf{x},t)$ the values of $\tilde{u}(\mathbf{x},t,\alpha)$ in the collocation points α_i and $h_i\left(\alpha\right)$ interpolating polynomials of degree $N_p - 1$, with $h_i(\alpha_j) = \delta_{ij}$. Transformation (12) is applied to the differential equation (1) after which expansion (13) is substituted. A Galerkin projection on each basis $\{h_l\}$ is applied to ensure that the error is orthogonal to the functions basis spanned by $\{h_i\}$:

$$\left\langle \mathcal{L} \sum_{i=1}^{N_p} \tilde{u}_i(\mathbf{x},t) h_i, h_l \right\rangle = \left\langle S, h_l \right\rangle, \qquad l = 1, \ldots, N_p. \tag{14}$$

The Galerkin projection (14) is approximated using Gaussian quadrature. For a general inner product $\langle f(\alpha), g(\alpha)\rangle$ of two functions $f(\alpha)$ and $g(\alpha)$ Gaussian quadrature results in:

$$\begin{aligned} \langle f(\alpha), g(\alpha)\rangle &= \sum_{i=1}^{N_p}\sum_{j=1}^{N_p}\sum_{l=1}^{N_p} f_i g_j h_i(\alpha_l) h_j(\alpha_l) w_l, \\ &= \sum_{i=1}^{N_p}\sum_{j=1}^{N_p}\sum_{l=1}^{N_p} f_i g_j \delta_{il}\delta_{jl} w_l, \\ &= \sum_{l=1}^{N_p} f_l g_l w_l, \end{aligned} \tag{15}$$

where w_l are the quadrature weights corresponding to the collocation points α_l. The resulting set of equations is fully decoupled. The mean μ_u and the variance σ_u^2 of the stochastic solution can be determined using:

$$\mu_u = \sum_{i=1}^{N_p} \tfrac{1}{2} u_i(\mathbf{x},t) w_i, \tag{16}$$

$$\sigma_u^2 = \sum_{i=1}^{N_p} \tfrac{1}{2}\left(u_i(\mathbf{x},t)\right)^2 w_i - \left(\sum_{i=1}^{N_p} \tfrac{1}{2} u_i(\mathbf{x},t) w_i\right)^2, \tag{17}$$

where w_i are the weights corresponding to the collocation points α_i. These relations are derived from the definition of the mean and variance.

There are other possibilities to choose the collocation points. In this paper the choice of Chebyshev points, given by Eq. (18), is explored. The originally proposed Gauss-Legendre points[18] are not performing well for distributions with tails like the exponential and normal distribution. That is why the choice is made to use Chebyshev points. They are more dense on the edges of $[-1,1]$ and therefore expected to be more suitable for approximation of the exponential and normal distribution, see Figure 1. Instead of the originally proposed Gauss-Legendre points combined with Lagrange interpolation, the Chebyshev approximation formula (Eqs. (19) and (20)) combined with Chebyshev points is used to approximate the stochastic response. Gaussian quadrature is used afterwards in order to obtain the integrals required for the computation of the mean and variance. The Chebyshev points are defined as

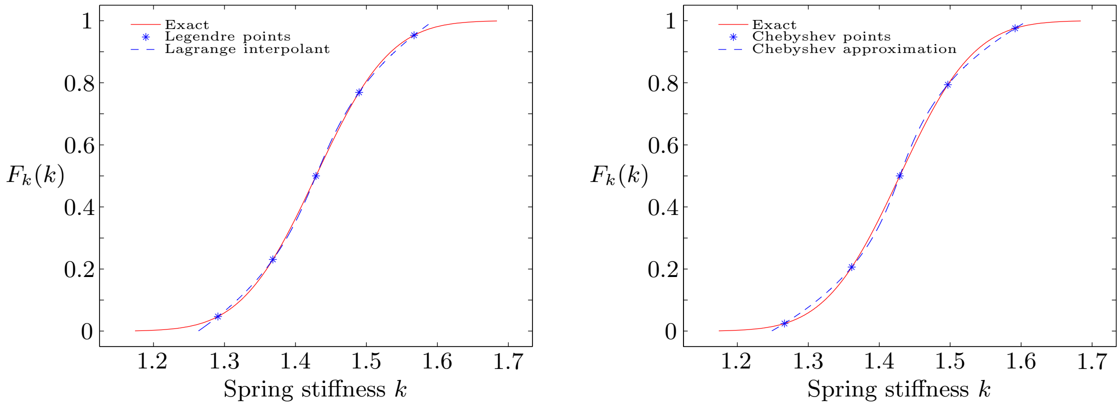
$$\alpha_i = \cos\left(\frac{\pi(i-\frac{1}{2})}{N_p}\right) \qquad i = 1, 2, \ldots, N_p. \tag{18}$$

The Chebyshev approximation formula is given by

$$c_j = \frac{2}{N}\sum_{k=1}^{N_p} f(\alpha_k)T_j(\alpha_k) \qquad j = 0, \ldots, N_p - 1, \tag{19}$$

$$f(x) \approx \sum_{k=0}^{N_p-1} c_k T_k(x) - \tfrac{1}{2}c_0, \tag{20}$$

where $T_j(x)$ are Chebyshev polynomials of the first kind.



(a) Legendre collocation points
(b) Chebyshev collocation points

Figure 1. Two different choices for collocation points and interpolating polynomials for the Stochastic Collocation method for a normally distributed spring stiffness k used in section IV for the linear piston problem.

III. Piecewise Interpolated Sampling

The Stochastic Collocation method uses global polynomials to approximate the stochastic response. Interpolation using global polynomials has an important disadvantage, namely the fact that a small fluctuation over a small portion of the interval can induce large fluctuations over the entire domain.[2] After long time integration distributions can be heavily deformed. Especially distributions with tails show large fluctuations in the tail areas. These local fluctuations decreases the accuracy of the global approximation. The error induced by these fluctuations is minimized and evenly spread over the domain using the Stochastic Collocation method. However, the collocation points are taken in the α domain, therefore the tail of the exponential distribution is very hard to approximate. Since only a few points are located in the tail of the distribution. In this paper the Piecewise Interpolated Sampling method is introduced which uses a uniform grid sampled from the parameter space. This is done to obtain a good coverage of the possible parameter values. A piecewise cubic interpolation is used to obtain an oscillation free approximation. When two samples are taken, the interpolation is linear. For three samples it is quadratic and for four samples or more cubic interpolation is used. When the M samples are taken from the parameter space, a minimum and maximum value for the parameter has to be set, since in tails the parameter value becomes $\pm\infty$. To be able to cope with this the distribution function of the input is chosen to be in the domain $[\epsilon, 1-\epsilon]$ with $\epsilon = M^{-2}$. This results in a good coverage of the parameter domain, also for a small number of grid points. The parts $[0, \epsilon]$ and $[1-\epsilon, 1]$ are approximated using extrapolation.

To show the difference between local and global polynomial approximation, the linear piston problem is used, see section IV. Figure 2 shows the approximation using 10 points with global (Stochastic Collocation) and local (Piecewise Interpolated Sampling) polynomial interpolation. The input distribution of the spring stiffness k is shown in Figure 2(a). As one can see, only a few collocation points are present in the tail of the distribution. The samples for the Piecewise Interpolated Sampling method are taken directly from the parameter domain, resulting in relatively many samples in the tail. Figure 2(b) shows the stochastic response of the piston position q at $t = 3$. Here it is clear that the slightly deformed distribution is hard to approximate using the Stochastic Collocation method since the tail is not approximated well. For a value of k larger than 1.45 the response q is not approximated well by the global polynomials, while the Piecewise Interpolated Sampling method obtains a good approximation.

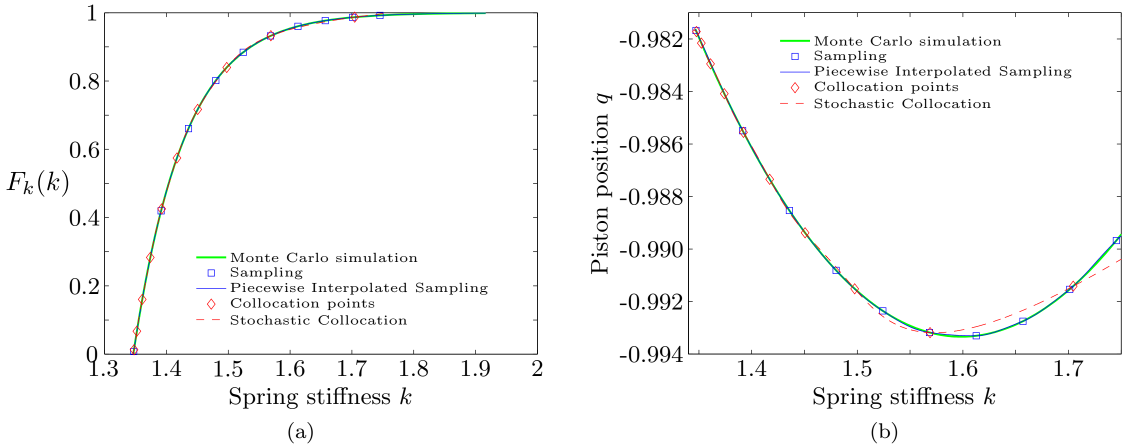


Figure 2. Approximation using 10 points and global (Stochastic Collocation) and local (Piecewise Interpolated Sampling) polynomial interpolation with (a) exponentially distributed spring stiffness k and (b) the stochastic response of the piston position q at $t = 3$.

Figure 3 shows the convergence of the relative error of the mean of the piston position μ_q with respect to the number of samples M. The Piecewise Interpolated Sampling method is here compared to other sampling methods. Here Monte Carlo simulation is included using random and stratified sampling. The convergence rate for the Monte Carlo simulation is as stated in section B, $\mathcal{O}(M^{-1/2})$ for random sampling and $\mathcal{O}(M^{-1})$ for stratified sampling. The Piecewise Interpolated Sampling has a convergence rate of $\mathcal{O}(M^{-1})$, but the accuracy of the approximation is 3 orders better.

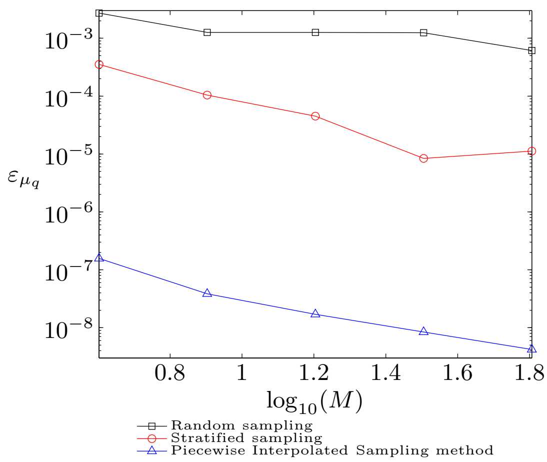


Figure 3. Convergence of the relative error of the mean of the piston position μ_q with respect to the number of samples M.

IV. Efficiency comparison of the methods for Step II

The efficiency of the uncertainty quantification methods for Step II (i.e. Monte Carlo simulation, Polynomial Chaos, Stochastic Collocation and Piecewise Interpolated Sampling) is investigated for a simple fluid-structure interaction problem, i.e. a linear piston problem.[16] Here the spring stiffness k is assumed to be uncertain with a uniform, exponential and normal probability distribution function. These three distribution are chosen since they respresent a different parameter domain: finite $[a, b]$, semi-infinite $[a, \infty)$ and infinite $(-\infty, \infty)$. For all three distributions the mean and variance are the same for an equal comparison. The methods are compared with respect to the error convergence and the amount of computational work. The work is expressed as the number of times a deterministic problem needs to be solved. For the Polynomial Chaos method the amount of work is $I * (P + 1)$ where P is the order of the approximation and I the number of Block-Gauss-Seidel iterations required to obtain the polynomial coefficients with an accuracy of 10^{-8}. For the following computations about 3-5 iterations were required. The three distribution types are all in the Askey scheme yielding an optimal polynomial chaos respresentation. The Stochastic Collocation method uses N_p deterministic computations, one for every collocation point. The Piecewise Interpolated Sampling method requires the number of samples M deterministic computations. As Lin et al.[13] stated the performance of the Askey Polynomial Chaos depends strongly on the length of the time integration. Therefore, the efficiency of all methods is investigated for short and long time integration.

A. The linear piston problem

The fluid-structure interaction test problem is the linear piston[16] indicated in Figure 4. The linear piston is chosen since it is a test problem which is easily evaluated, but still has all aspects of fluid-structure interaction. It consists of a one-dimensional fluid domain bounded by a mass which is attached to a spring.

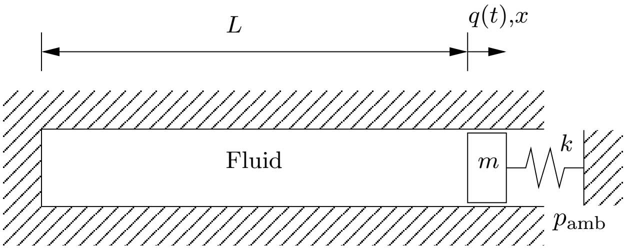


Figure 4. Linear piston problem.

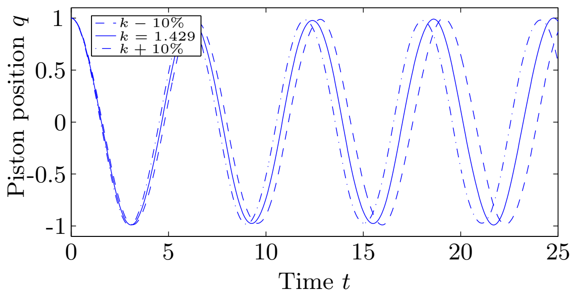


Figure 5. Realizations using different values for the spring stiffness k of the linear piston problem for $k = \mu_k = 1.429$ and $k = \mu_k \pm 10\%$.

The fluid is modeled using the linearized isentropic Euler equations. The piston position $q(t)$ and velocity $\dot{q}(t)$ are determined by the mass of the piston m, the spring stiffness k, the length of the fluid domain L, the fluid state $U = (\rho, \rho u)^T$ and the ambient pressure p_{amb}. The problem is solved for small deviations from the equilibrium only. Therefore the fluid state vector can be split into two parts

$$U = U_0 + U', \tag{21}$$

with U_0 the equilibrium state and U' the small deviation. The governing equations for the deviations become:

$$\frac{\partial U'}{\partial t} = -\begin{bmatrix} 0 & 1 \\ c_0^2 & 0 \end{bmatrix} \frac{\partial U'}{\partial x}, \tag{22}$$

with $c = \sqrt{\frac{\partial p}{\partial \rho}}$ the speed of sound. The structure is governed by:

$$m\ddot{q} + kq = p_{x=L} - p_{\text{amb}}, \tag{23}$$

where q is the displacement of the piston, $p_{x=L}$ is the pressure acting from the fluid on the piston and p_{amb} is the ambient pressure surrounding the piston. The flow and structure Eqs. (22) and (23) are made dimensionless using the following substitutions

$$\bar{t} = \frac{c_0}{L}t,\ \bar{x} = \frac{x}{L},\ \bar{q} = \frac{q}{L},\ \bar{\rho} = \frac{\rho}{\rho_0},\ \bar{u} = \frac{u}{c_0},\ \bar{m} = \frac{m}{\rho_0 L},\ \bar{k} = \frac{kL}{\rho_0 c_0^2} \text{ and } \bar{U} = \begin{bmatrix} \bar{\rho} \\ \bar{\rho}\bar{u} \end{bmatrix}.$$

The resulting non-dimensionless set of equations becomes:

$$\frac{\partial \bar{U}'}{\partial \bar{t}} = -\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{\partial \bar{U}'}{\partial \bar{x}}, \tag{24}$$

$$\bar{m}\ddot{\bar{q}}' + \bar{k}\bar{q}' = \bar{\rho}'. \tag{25}$$

The coupling at the boundary $\bar{x} = 1$ is given by:

$$\bar{u}'\left(\bar{x} = 1\right) = \dot{\bar{q}}'. \tag{26}$$

The other boundary condition for the fluid is a fixed wall

$$\bar{u}'\left(\bar{x} = 0\right) = 0. \tag{27}$$

From now on the bar and prime to indicate non-dimensionality and small deviations are omitted for convenience of notation. Equation (24) is discretized using a second-order central finite volume discretization. Together with Eq. (25) the equations are written as the linear system

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = A\mathbf{x}, \tag{28}$$

in which

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_\mathrm{f} \\ \mathbf{x}_\mathrm{s} \end{bmatrix} \text{ and } A = \begin{bmatrix} A_\mathrm{f} & A_\mathrm{fs} \\ A_\mathrm{sf} & A_\mathrm{s} \end{bmatrix},$$

where $\mathbf{x}_\mathrm{f}$ contains the fluid properties U_i of all finite volume cells and $\mathbf{x}_\mathrm{s}$ contains the piston position q and the piston velocity $\dot{q}$. The matrices A_fs and A_sf provide the coupling between the fluid and the structure. The time integration is performed using an ESDIRK-4 scheme.[11,27]

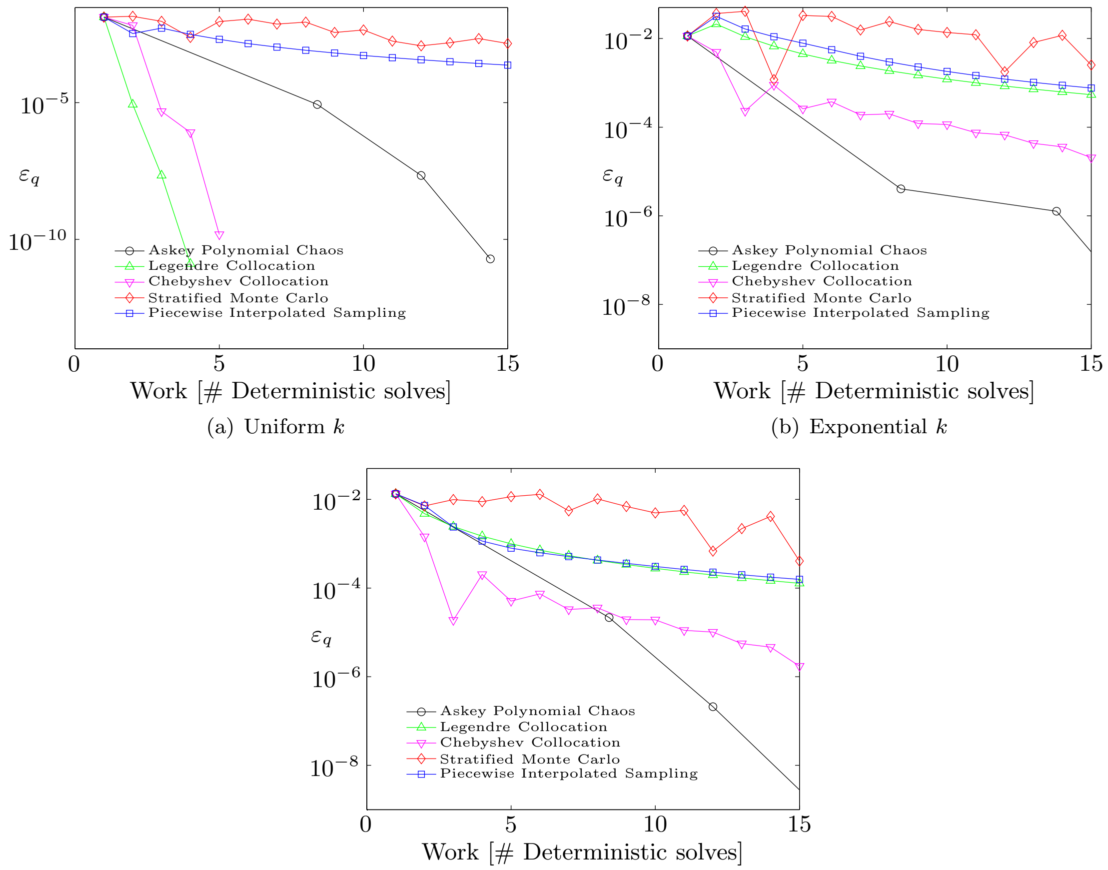
B. Efficiency for short time integration

In this section short time integration is considered. The piston position is evaluated till $T = 10$ which corresponds to approximately 1.5 period. The stochastic response is computed using Monte Carlo simulation with stratified sampling, Askey Polynomial Chaos, Stochastic Collocation with Gauss-Legendre points and Chebyshev points and the Piecewise Interpolated Sampling method. The error convergence is considered of the relative error of the mean of the piston position q, which is defined as

$$\varepsilon_q(T) = \frac{\mu_q(T) - \mu_{q,\text{reference}}(T)}{\mu_{q,\text{reference}}(T)}. \tag{29}$$

The reference solution is computed using a 20^{th} order Askey Polynomial Chaos approximation. The relative error of the mean with respect to the amount of computational work is indicated in Figures 6(a) till (c). From the figures the following can be concluded.

Uniformly distributed input For a uniformly distributed input parameter clearly the Legendre Collocation is the most efficient method, see Figure 6(a). The convergence with respect to polynomial order is the same for the Legendre Collocation and the Askey Polynomial Chaos method. However, the amount of deterministic solves is for the Legendre Collocation equal to the number of collocation points N_p, while for the Askey Polynomial Chaos it is equal to the number of terms times the number of Block-Gauss-Seidel iterations $I * (P + 1)$. For the Chebyshev Collocation the odd orders are holding up the convergence. This is because the effect of odd and even functions is visible in the Chebyshev approximation. The Piecewise Interpolated Sampling method is converging slightly better than the Monte Carlo simulation, the accuracy is about one order better with the same amount of work. For short time integration and uniformly distributed input parameters the Legendre Collocation is the most efficient method.



(a) Uniform k

(b) Exponential k

(c) Normal k

Figure 6. Error convergence of the mean of the piston position μ_q at $T = 10$ (approximately 1.5 period) with respect to the amount of computational work, expressed in the number of times a deterministic system has to be solved.

Exponentially distributed input An exponentially distributed input parameter shows a very different picture, see Figure 6(b). Here the Askey Polynomial Chaos is the most efficient method, although the convergence is not smooth. The Chebyshev Collocation here performes significantly better than the original Legendre Collocation. The Chebyshev points are more dense in the tail of the exponential distribution and therefore the Chebyshev polynomials results in a better approximation. The Piecewise Interpolated Sampling method shows the same convergence as the Legendre Collocation. Both are one order more accurate with the same amount of work than Monte Carlo simulation. In conclusion the Askey Polynomial Chaos is the most efficient method for an exponentially distributed input distribution for short time integration.

Normally distributed input When the input parameter is normally distributed two methods are efficient. Depending on the required accuracy of the computation one of the methods is the most efficient, see Figure 6(c). For relatively low accuracy, an error larger than 10^{-5}, the Chebyshev Collocation is more efficient. When a high accuracy is required, however, the Askey Polynomial Chaos is the most efficient method. The turning point is at approximately 8 deterministic solves. Again the Legendre Collocation and the Piecewise Interpolated Sampling method show the same convergence, about one order more accurate than the Monte Carlo simulation with the same amount of work. For a normally distributed input parameter the Chebyshev Collocation is therefore the most efficient method for low accuracy and the Askey Polynomial Chaos is the best choice when high accuracy is required.

C. Efficiency for long time integration

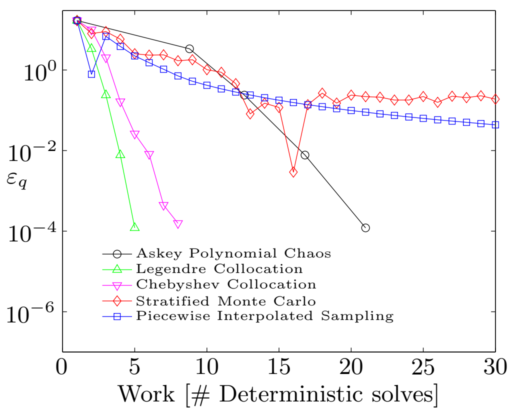
The long time integration is investigated by evaluating the piston position till $T = 100$, which corresponds to approximately 15 periods. The reference solution for this case is computed using Monte Carlo simulation with 10^6 samples. Figures 7(a) till (c) show the convergence of the relative error of the mean of the piston position q with respect to the amount of work.

Uniformly distributed input For a uniformly distributed input parameter the Legendre Collocation is the most efficient method, see Figure 7(a), just as for short time integration. The Legendre Collocation shows the same convergence with respect to polynomial order as the Askey Polynomial Chaos, but the amount of work is less. The Chebyshev Collocation is again limited by the odd orders. The Piecewise Interpolated Sampling method performs slightly better than the Monte Carlo simulation. For the uniformly distributed input parameters, the Legendre Collocation is the most efficient for long time integration.

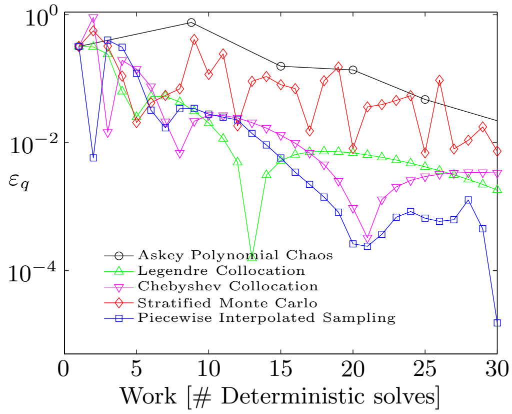
Exponentially distributed input For an exponentially distributed input parameter the convergence of all methods is not smooth due to the large deformed tail of the distribution, see Figure 7(b). Now till an accuracy of 10^{-2} no conclusions can be drawn. For higher accuracy the Piecewise Interpolated Sampling method performs best. The Askey Polynomial Chaos shows good convergence with respect to polynomial order, however in this figure it is not clear since it takes much more work. The deformed tail proved to be very hard to approximate using global polynomials for both collocation methods. For the exponentially distributed input parameter the Piecewise Interpolated Sampling method is therefore the most efficient for long time integration.

Normally distributed input As for the short time integration two methods can be chosen depending on the requirements of the computation, see Figure 7(c). The Chebyshev Collocation is the most efficient method for accuracies up to $2{*}10^{-3}$. For more accurate approximations the Askey Polynomial Chaos method is the best choice. However, one have to take in mind that the amount of work is already 30 deterministic solves at that point. The Lagrange Collocation and the Piecewise Interpolated Sampling method are performing better than the Monte Carlo simulation, both obtain 2 orders more accuracy using the same amount of work. For long time intergration the Chebyshev Collocation is most efficient for an accuracy till $2 * 10^{-3}$ and the Askey Polynomial Chaos for higher accuracies.

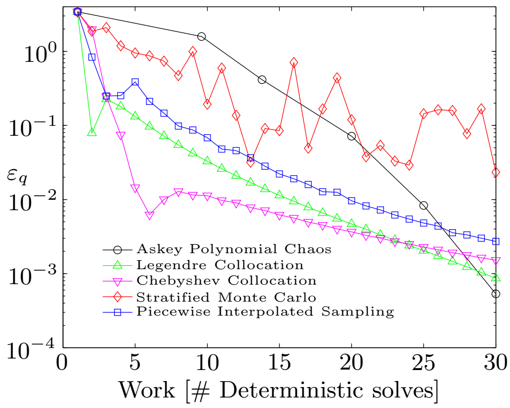
The results for short and long time integration do not show one method which is the most efficient in all cases. This means that for every case based on the required accuracy, length of the time integration and distribution of the input parameter a method has to be chosen which is most efficient for that case. It is likely to assume that in general the results will hold for variance approximations and general problems as well.



(a) Uniform k



(b) Exponential k



(c) Normal k

Figure 7. Error convergence of the mean of the piston position μ_q at $T = 100$ (approximately 15 periods) with respect to the amount of computational work, expressed in the number of times a deterministic system has to be solved.

V. Demonstration of the Two Step approach for the linear piston problem

To demonstrate the Two Step approach for uncertainty quantification, the linear piston problem is used with a moving wall at the left boundary, see Figure 8. The forcing is assumed to be harmonic according $f(t) = A\sin(\omega t)$, with A the forcing amplitude and ω the forcing frequency. The parameters are set to

$$k = 1 \qquad m = 1 \qquad A = 0.1 \qquad \omega = 0.8.$$

These four parameters are assumed to be uncertain, and uniformly distributed according $U(\mu - 10\%, \mu + 10\%)$. The forcing frequency is set here to $\omega = 0.8$ since the solution is not diverging for this value. Taking ω close to one leads to a diverging oscillation.

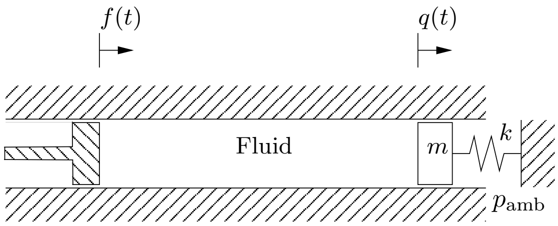
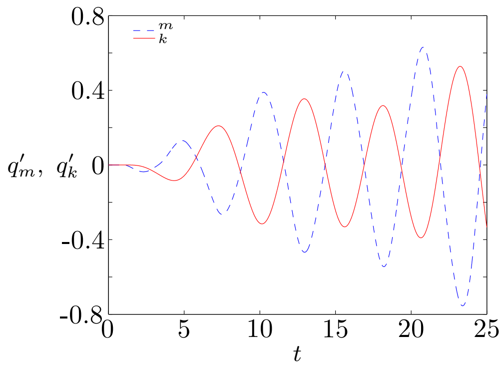


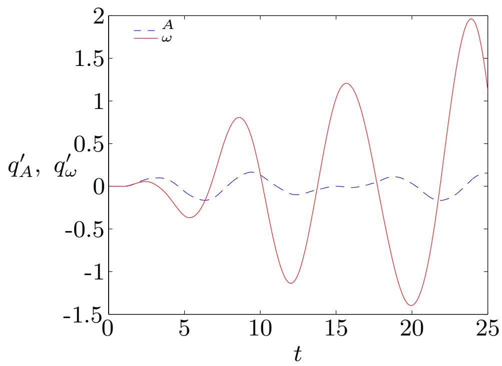
Figure 8. Linear piston with an unsteady forcing $f(t)$ at the left wall.

A. Step I: Identifying the most important parameter for the piston problem using Sensitivity Analysis

A Sensitivity Analysis is performed for the forcing amplitude A, forcing frequency ω, spring stiffness k and the piston mass m. The computational cost of this first step is equal to 5 deterministic solves.



(a)



(b)

Figure 9. Sensitivity derivatives of the piston position with respect to all dependent parameters. (a) the structure parameters, piston mass m and spring stiffness k. (b) the boundary condition parameters, the forcing amplitude A and the forcing frequency ω.

Figures 9(a) and (b) show the scaled sensitivity derivatives with respect to each parameter. Figure 9(a) gives the sensitivity with respect to the structure parameters, which are the piston mass m and the spring stiffness k. Both sensitivity derivatives diverge in time, therefore k and m have an increasing influence on the solution in time. Figure 9(b) presents the parameters in the boundary condition, the forcing amplitude A and the forcing frequency ω. From this figure it can be concluded that the frequency ω has a much larger influence on the solution than the amplitude A. The sensitivity derivative with respect to the amplitude A does not diverge in time, while the derivative with respect to the frequency ω diverges very fast. For longer time integration the amplitude A does not have a significant influence on the solution. The sensitivity derivatives of the two structural parameters k and m also diverge, however, not as fast as the forcing frequency ω. Therefore the next section presents the results for the uncertainty quantification for an uncertain forcing frequency ω.

B. Step II: Obtaining the stochastic response of the piston position q using the Stochastic Collocation method

The forcing frequency is assumed to be uniformly distributed by $U(\mu_\omega - 10\%, \mu_\omega + 10\%)$ with mean $\mu_\omega = 0.8$, resulting in a variation around the mean of 10 percent. Since ω occurs in non-polynomial terms in the equations as for instance $\omega \cos(\omega t)$ the generalized Polynomial Chaos method is not straightforward. The previous sections showed that for a uniformly distributed parameter the Legendre Collocation method is the most efficient method for both short and long time integration. The non-polynomial terms are not a problem for the Stochastic Collocation method.

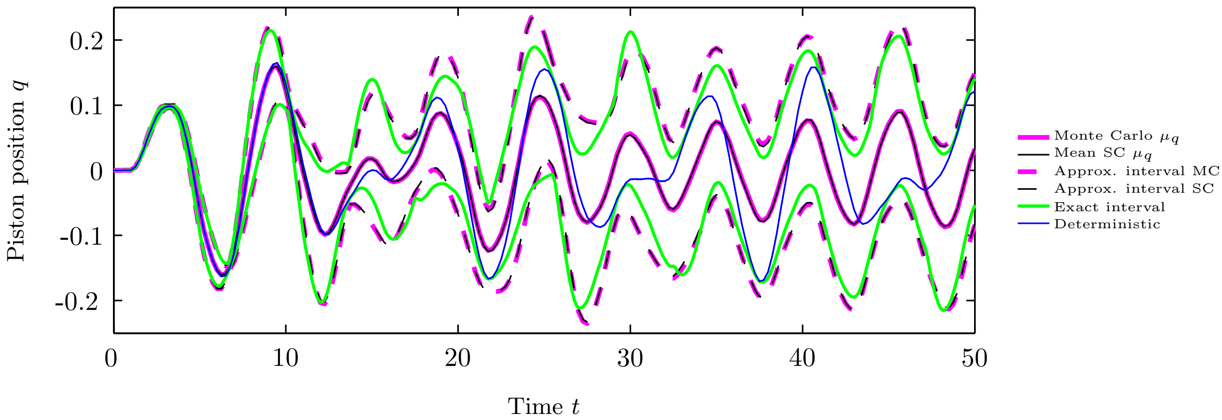


Figure 10. The piston position response for $t = [0, 50]$. The deterministic response for $\omega = \mu_\omega$ is included as well as the mean response μ_q and the approximated interval $[\mu_q - \sqrt{3}\sigma_q, \mu_q + \sqrt{3}\sigma_q]$ based on a uniformly distributed random variable.

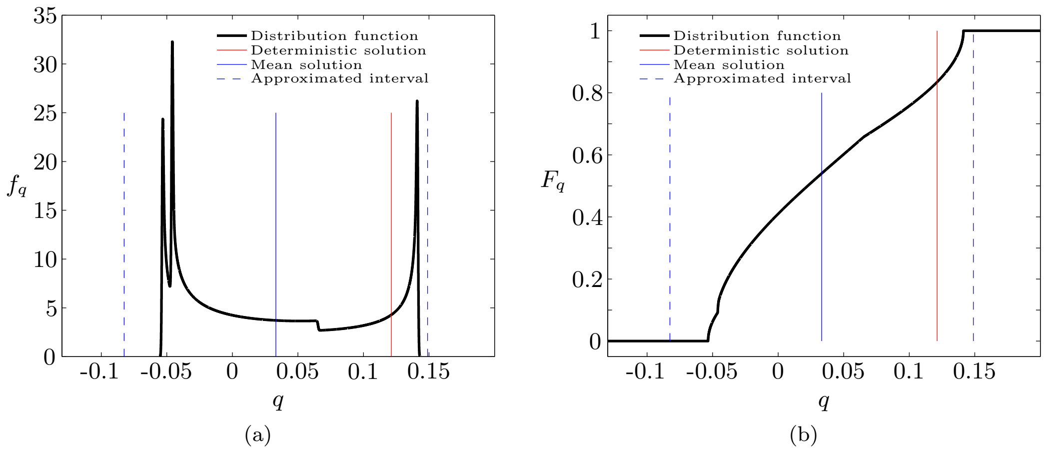


Figure 11. The stochastic properties of the piston position q at $t = 50$. The probability density function $f_q(q)$ (a) and the probability distribution function $F_q(q)$ (b). The deterministic result for $\omega = \mu_\omega$ and an approximated interval $[\mu_q - \sqrt{3}\sigma_q, \mu_q + \sqrt{3}\sigma_q]$ based a uniformly distributed random variable are included.

The response of the piston position q is shown in Figure 10 for $t \in [0, 50]$. The deterministic solution is included as well as the results for a Monte Carlo simulation using 500 samples. A 7^{th} order approximation is obtained efficiently using the Stochastic Collocation method. The exact interval which contains all possible values for the piston position q is obtained from the Monte Carlo simulation. The interval is approximated by $[\mu_q - \sqrt{3}\sigma_q, \mu_q + \sqrt{3}\sigma_q]$, it is exact for a uniform distribution. Due to the time integration the distribution is not uniform and not even symmetric anymore. The approximated interval is symmetric, that is why in Figure 10 at some points the real interval is outside the approximated interval. When the mean and variance are used to approximate an interval based on the input distribution, the resulting interval can differ significantly from the real interval. This can best be seen in Figure 10, at $t = 27$ the interval is too conservative, while at $t = 30$ the approximated interval is too small.

Figure 11 shows the probability distribution of the piston position q at $t = 50$ resulting from a uniformly distributed spring stiffness k. Figure 11(a) shows the probability density function $f_q(q)$ and Figure 11(b) shows the probability distribution function $F_q(q)$ of the piston position q. The figure perfectly shows the importance of computing the distribution function $F_q(q)$ instead of only the mean and variance. The mean value has a small probability to occur, just as the deterministic solution. From Figure 11(a), it can be seen that the probability of the mean is equal to the probability of the deterministic solution for this case. The highest probability of the solution is at the edges of the domain. The piston position at $t = 50$ is most likely to be $q(50) = -0.04$ or $q(50) = 0.14$ at the peaks of the probability density function $f_q(q)$. This is a huge difference with both the mean and the deterministic solution. This result perfectly illustrates the importance of uncertainty quantification.

C. Efficiency of the Two Step approach

The use of the Two Step approach increased the efficiency of the uncertainty quantification significantly. Table 1 shows an overview of the amount of computational work required to obtain the stochastic response of the piston position q based on the four uncertain input parameters. When the stochastic response is computed for all uncertain parameters, the Stochastic Collocation (Legendre) is the most efficient method. However, it requires still 4096 deterministic solves. This is unacceptable, since deterministic computational fluid-structure interaction computations are extremely computationally demanding. The Two Step approach requires 5 deterministic solves to identify the most important parameter using Sensitivity Analysis, after which the Stochastic Collocation method provides a 7^{th} order approximation of the stochastic response of the piston position requiring 8 deterministic solves. Consequently, the Two Step approach uses only 13 deterministic solves in total. When the Askey Polynomial Chaos method was used for Step II, 37 deterministic solves were required. This is due to the fact that 4 Block-Gauss-Seidel iterations were necessary to obtain the polynomial coefficients with an accuracy of 10^{-8}.

The maximal gain in efficiency of the Two Step approach obtains a speed-up factor of $\mathcal{O}(10^9)$ with respect to Monte Carlo simulation for all four parameters. When for all parameters the Stochastic Collocation method is employed, still a speed-up factor of $\mathcal{O}(10^2)$ is obtained by the Two Step approach. Here, one must be aware that including all four parameters in the uncertainty quantification provides more information about the solution and combined effects of uncertain parameters. However, the amount of work required for a full four parameter uncertainty quantification is unacceptable for computational fluid-structure interaction problems.

VI. Conclusions

In this paper a Two Step approach is followed for efficient uncertainty quantification in computational fluid-structure interaction problems with multiple uncertain parameters. The first step consists of a Sensitivity Analysis to identify the most important parameter of the problem. This is an efficient way to reduce the problem to one uncertain parameter. In the second step a more advanced method is employed to obtain the stochastic response based on the probability distribution of the uncertain input parameter. For this the efficiency of Monte Carlo simulation, the Askey Polynomial Chaos method, and the Stochastic Collocation method, is compared. In addition, the Piecewise Interpolated Sampling method for the second step was presented and included in the comparison on efficiency.

Three input distributions were used: uniform, exponential and normal. The comparison showed that not a single method can be identified which is the most efficient in all cases. Only for a uniformly distributed input parameter the Legendre Stochastic Collocation is the most efficient for both short and long time integration. The Askey Polynomial Chaos method shows exponential convergence with respect to the polynomial order. However, due to the coupled system of equations that has to be solved the amount of work is a factor 3–5 higher than the number of terms included in the expansion. When high accuracy is required for short time integration and exponentially and normally distributed input parameters the Askey Polynomial Chaos method is most efficient. For long time integration the Askey Polynomial Chaos method for exponentially

Table 1. Overview of the amount of computational work expressed in the number of times a deterministic system is solved for the linear piston problem with an unsteady boundary condition. Four parameters are assumed to be uniformly distributed.

Approach	(Step II) Method	Sensitivity Analysis [det. solves]	Method to obtain the stochastic response of q [det. solves]	Total amount of Work [det. solves]
All four parameters	7^{th} order Stochastic Collocation	-	8^4	4096
	7^{th} order Askey Polynomial Chaos	-	$(4+7)!/(7!4!) \times 4$	5280
	Monte Carlo simulation using 500 samples	-	500^4	$6.25 \cdot 10^{10}$
Two Step	7^{th} order Stochastic Collocation	5	8	13
	7^{th} order Askey Polynomial Chaos	5	$8 \times 4 = 32$	37
	Monte Carlo simulation using 500 samples	5	500	505

and normally distributed input parameters

The Two Step approach has been demonstrated for the linear piston problem with an unsteady boundary condition. First in Step I Sensitivity Analysis identified the forcing frequency ω as the most important parameter for this case. Since the forcing frequency is uniformly distributed the stochastic response of the piston position q is obtained in Step II using the Legendre Stochastic Collocation method. A 7^{th} order approximation is computed with 8 collocation points. The results show how uncertainty quantification is used to obtain reliable results. Also the importance is clear why the complete stochastic response of the solution is required and not only approximations of the mean and variance. The amount of work significantly decreased using the Two Step approach: only 13 deterministic solves were required for the 7^{th} order approximation. When all four parameters would have been included 4096 deterministic solves are needed to obtain the stochastic response with the Legendre Stochastic Collocation method. As a result, the Two Step approach is an efficient way to include uncertainties of the input parameters in the problem. A reduction factor of $\mathcal{O}(10^2)$ is obtained compared to the full uncertainty analysis of all uncertain parameters.

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